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# $L^1$ - CONVERGENCE OF GREEDY ALGORITHM BY GENERALIZED WALSH SYSTEM

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**ABSTRACT.** In this paper we consider the generalized Walsh system and a problem  $L^1$  – *convergence* of greedy algorithm of functions after changing the values on small set .

## 1. INTRODUCTION AND PRELIMINARIES

Let  $a$  denote a fixed integer,  $a \geq 2$  and put  $\omega_a = e^{\frac{2\pi i}{a}}$ . Now we will give the definitions of generalized Rademacher and Walsh systems [1].

**Definition 1.1.** The Rademacher system of order  $a$  is defined by

$$\varphi_0(x) = \omega_a^k \text{ if } x \in \left[ \frac{k}{a}, \frac{k+1}{a} \right), \quad k = 0, 1, \dots, a-1, \quad x \in [0, 1)$$

and for  $n \geq 0$

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x).$$

**Definition 1.2.** The generalized Walsh system of order  $a$  is defined by

$$\psi_0(x) = 1,$$

and if  $n = \alpha_1 a^{n_1} + \dots + \alpha_s a^{n_s}$  where  $n_1 > \dots > n_s$ , then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdot \dots \cdot \varphi_{n_s}^{\alpha_s}(x).$$

Let's denote the generalized Walsh system of order  $a$  by  $\Psi_a$ .

Note that  $\Psi_2$  is the classical Walsh system.

The basic properties of the generalized Walsh system of order  $a$  are obtained by H.E.Chrestenson, R. Pely, J. Fine, W. Young, C. Vatari, N. Vilenkin and others (see [1]- [7]).

In this paper we consider  $L^1$ - convergence of greedy algorithm with respect to  $\Psi_a$  system. Now we present the definition of greedy algorithm.

Let  $X$  be a Banach space with a norm  $\|\cdot\| = \|\cdot\|_X$  and a basis  $\Phi = \{\phi_k\}_{k=1}^\infty$ ,  $\|\phi_k\|_X = 1$ ,  $k = 1, 2, \dots$ .

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For a function  $f \in X$  we consider the expansion

$$f = \sum_{k=1}^{\infty} a_k(f) \phi_k \quad .$$

**Definition 1.3.** Let an element  $f \in X$  be given. Then the  $m$ -th greedy approximant of the function  $f$  with regard to the basis  $\Phi$  is given by

$$G_m(f, \phi) = \sum_{k \in \Lambda} a_k(f) \phi_k,$$

where  $\Lambda \subset \{1, 2, \dots\}$  is a set of cardinality  $m$  such that

$$|a_n(f)| \geq |a_k(f)|, \quad n \in \Lambda, \quad k \notin \Lambda.$$

In particular we'll say that the greedy approximant of  $f \in L^p[0, 1]$ ,  $p \geq 0$  converges with regard to the  $\Psi_a$ , if the sequence  $G_m(x, f)$  converges to  $f(t)$  in  $L^p$  norm. This new and very important direction invaded many mathematician's attention (see [8]-[16]).

T.W. Körner [10] constructed an  $L^2$  function (then a continuous function) whose greedy algorithm with respect to trigonometric systems diverges almost everywhere.

V.N.Temlyakov in [11] constructed a function  $f$  that belongs to all  $L^p$ ,  $1 \leq p < 2$  (respectively  $p > 2$ ), whose greedy algorithm concerning trigonometric systems divergence in measure (respectively in  $L^p$ ,  $p > 2$ ), e.i. the trigonometric system are not a quasi-greedy basis for  $L^p$  if  $1 < p < \infty$ .

In [13] R.Gribonval and M.Nielsen proved that for any  $1 < p < \infty$  there exists a function  $f(x) \in L^p[0, 1]$  whose greedy algorithm with respect to  $\Psi_2$ - classical Walsh system diverges in  $L^p[0, 1]$ . Moreover, similar result for  $\Psi_a$  system follows from Corollary 2.3. (see [13]). Note also that in [15] and [16] this result was proved for  $L^1[0, 1]$ .

The following question arises naturally: is it possible to change the values of any function  $f$  of class  $L^1$  on small set, so that a greedy algorithm of new modified function concerning  $\Psi_a$  system converges in the  $L^1$  norm?

The classical **C**-property of Luzin is well-known, according to which every measurable function can be converted into a continuous one by changing it on a set of arbitrarily small measure. This famous result of Luzin [17] dates back to 1912.

Note that Luzin's idea of modification of a function improving its properties was substantially developed later on.

In 1939, Men'shov [18] proved the following fundamental theorem.

**Theorem (Men'shov's C-strong property).** *Let  $f(x)$  be an a.e. finite measurable function on  $[0, 2\pi]$ . Then for each  $\varepsilon > 0$  one can define a continuous function  $g(x)$  coinciding with  $f(x)$  on a subset  $E$  of measure  $|E| > 2\pi - \varepsilon$  such that its Fourier series with respect to the trigonometric system converges uniformly on  $[0, 2\pi]$ .*

Further interesting results in this direction were obtained by many famous mathematicians (see for example [19]-[23]).

Particularly in 1991 M. Grigorian obtain the following result [20]:

**Theorem ( $L^1$ -strong property).** *For each  $\varepsilon > 0$  there exists a measurable set  $E \subset [0, 2\pi]$  of measure  $|E| > 2\pi - \varepsilon$  such that for any function  $f(x) \in L^1[0, 2\pi]$  one can find a function  $g(x) \in L^1[0, 2\pi]$  coinciding with  $f(x)$  on  $E$  so that its Fourier series with respect to the trigonometric system converges to  $g(x)$  in the metric of  $L^1[0, 2\pi]$ .*

In this paper we prove the following:

**Theorem 1.4.** *For any  $\varepsilon \in (0, 1)$  and for any function  $f \in L^1[0, 1)$  there is a function  $g \in L^1[0, 1)$ , with  $\text{mes}\{x \in [0, 1) ; g \neq f\} < \varepsilon$ , such that the nonzero fourier coefficients by absolute values monotonically decreasing.*

**Theorem 1.5.** *For any  $0 < \varepsilon < 1$  and each function  $f \in L^1[0, 1)$  one can find a function  $g \in L^1[0, 1)$ ,  $\text{mes}\{x \in [0, 1) ; g \neq f\} < \varepsilon$ , such that its fourier series by  $\Psi_a$  system  $L^1$  convergence to  $g(x)$  and the nonzero fourier coefficients by absolute values monotonically decreasing, i.e. the greedy algorithm by  $\Psi_a$  system  $L^1$ -convergence.*

The Theorems 1.1 and 1.2 follows from next more general Theorem 1.3, which in itself is interesting:

**Theorem 1.6.** *For any  $0 < \varepsilon < 1$  there exists a measurable set  $E \subset [0, 1)$  with  $|E| > 1 - \varepsilon$  and a series by  $\Psi_a$  system of the form*

$$\sum_{i=1}^{\infty} c_i \psi_i(x), \quad |c_i| \downarrow 0$$

*such that for any function  $f \in L^1[0, 1)$  one can find a function  $g \in L^1[0, 1)$ ,*

$$g(x) = f(x); \quad \text{if } x \in E$$

*and the series of the form*

$$\sum_{n=1}^{\infty} \delta_n c_n \psi_n(x), \quad \text{where } \delta_n = 0 \text{ or } 1,$$

*which convergence to  $g(x)$  in  $L^1[0, 1)$  metric and*

$$\left\| \sum_{n=1}^m \delta_n c_n \psi_n(x) \right\|_1 \leq 12 \cdot \|f\|_1, \quad \forall m \geq 1.$$

*Remark 1.7.* Theorems 1.6 for classical Walsh system  $\Psi_2$  was proved by M. Grigorian [21].

*Remark 1.8.* From Theorem 1.5 follows that generalized Walsh system  $\Psi_a$  has  $L^1$ -strong property.

## 2. BASIC LEMMAS

First we present some properties of  $\Psi_a$  system (see Definition 1.2).

**Property 1.** Each  $n$ th Rademacher function has period  $\frac{1}{a^n}$  and

$$\varphi_n(x) = \text{const} \in \Omega_a = \{1, \omega_a, \omega_a^2, \dots, \omega_a^{a-1}\}, \quad (2.1)$$

if  $x \in \Delta_{n+1}^{(k)} = [\frac{k}{a^{n+1}}, \frac{k+1}{a^{n+1}})$ ,  $k = 0, \dots, a^{n+1} - 1$ ,  $n = 1, 2, \dots$ .

It is also easily verified, that

$$(\varphi_n(x))^k = (\varphi_n(x))^m, \quad \forall n, k \in \mathcal{N}, \text{ where } m = k \pmod{a} \quad (2.2)$$

**Property 2.** It is clear, that for any integer  $n$  the Walsh function  $\psi_n(x)$  consists of a finite product of Rademacher functions and accepts values from  $\Omega_a$ .

**Property 3.** Let  $\omega_a = e^{\frac{2\pi i}{a}}$ . Then for any natural number  $m$  we have

$$\sum_{k=0}^{a-1} \omega_a^{k \cdot m} = \begin{cases} a, & \text{if } m \equiv 0 \pmod{a}, \\ 0, & \text{if } m \not\equiv 0 \pmod{a}. \end{cases} \quad (2.3)$$

**Property 4.** The generalized Walsh system  $\Psi_a$ ,  $a \geq 2$  is a complete orthonormal system in  $L^2[0, 1)$  and basis in  $L^p[0, 1)$ ,  $p > 1$  [3].

**Property 5.** From definition 2 we have

$$\psi_i(x) \cdot \psi_j(a^s x) = \psi_{j \cdot a^s + i}(x), \quad \text{where } 0 \leq i, j < a^s, \quad (2.4)$$

and particularly

$$\psi_{a^k + j}(x) = \varphi_k(x) \cdot \psi_j(x), \quad \text{if } 0 \leq j \leq a^k - 1. \quad (2.5)$$

Now for any  $m = 1, 2, \dots$  and  $1 \leq k \leq a^m$  we put  $\Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$  and consider the following function

$$I_m^{(k)}(x) = \begin{cases} 1, & \text{if } x \in [0, 1) \setminus \Delta_m^{(k)}, \\ 1 - a^m, & \text{if } x \in \Delta_m^{(k)}, \end{cases} \quad (2.6)$$

and periodically extend these functions on  $R^1$  with period 1.

By  $\chi_E(x)$  we denote the characteristic function of the set  $E$ , i.e.

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases} \quad (2.7)$$

Then, clearly

$$I_m^{(k)}(x) = \psi_0(x) - a^m \cdot \chi_{\Delta_m^{(k)}}(x), \quad (2.8)$$

and for the natural numbers  $m \geq 1$  and  $1 \leq i \leq a^m$

$$a_i(\chi_{\Delta_m^{(k)}}) = \int_0^1 \chi_{\Delta_m^{(k)}}(x) \cdot \overline{\psi_i}(x) dx = \mathcal{A} \cdot \frac{1}{a^m}, \quad 0 \leq i < a^m. \quad (2.9)$$

$$b_i(I_m^{(k)}) = \int_0^1 I_m^{(k)}(x) \overline{\psi_i}(x) dx = \begin{cases} 0, & \text{if } i = 0 \text{ and } i \geq a^k, \\ -\mathcal{A}, & \text{if } 1 \leq i < a^k \end{cases} \quad (2.10)$$

where  $\mathcal{A} = \text{const} \in \Omega_a$  and  $|\mathcal{A}| = 1$ .

Hence

$$\chi_{\Delta_m^{(k)}}(x) = \sum_{i=0}^{a^k-1} b_i(\chi_{\Delta_m^{(k)}})\psi_i(x) , \quad (2.11)$$

$$I_m^{(k)}(x) = \sum_{i=1}^{a^k-1} a_i(I_m^{(k)})\psi_i(x) . \quad (2.12)$$

**Lemma 2.1.** *For any numbers  $\gamma \neq 0$ ,  $N_0 > 1$ ,  $\varepsilon \in (0, 1)$  and interval by order  $a$   $\Delta = \Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$ ,  $i = 1, \dots, a^m$  there exists a measurable set  $E \subset \Delta$  and a polynomial  $P(x)$  by  $\Psi_a$  system of the form*

$$P(x) = \sum_{k=N_0}^N c_k \psi_k(x)$$

which satisfy the conditions:

$$1) \quad \text{coefficients } \{c_k\}_{k=N_0}^N \text{ equal } 0 \text{ or } -\mathcal{K} \cdot \gamma \cdot |\Delta|,$$

where  $\mathcal{K} = \text{const} \in \Omega_a$ ,  $|\mathcal{K}| = 1$ ,

$$2) \quad |E| > (1 - \varepsilon) \cdot |\Delta|,$$

$$3) \quad P(x) = \begin{cases} \gamma, & \text{if } x \in E; \\ 0, & \text{if } x \notin \Delta. \end{cases}$$

$$4) \quad \frac{1}{2} \cdot |\gamma| \cdot |\Delta| < \int_0^1 |P(x)| dx < 2 \cdot |\gamma| \cdot |\Delta|.$$

$$5) \quad \max_{N_0 \leq m \leq N} \int_0^1 \left| \sum_{k=N_0}^m c_k \psi_k(x) \right| < a \cdot |\gamma| \cdot \sqrt{\frac{|\Delta|}{\varepsilon}}.$$

*Proof.* We take a natural numbers  $\nu_0$  s so that

$$\nu_0 = \left\lceil \log_a \frac{1}{\varepsilon} \right\rceil + 1; \quad s = [\log_a N_0] + m. \quad (2.13)$$

Define the coefficients  $c_n$ ,  $a_i$ ,  $b_j$  and the function  $P(x)$  in the following way:

$$P(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}(a^s x), \quad x \in [0, 1] , \quad (2.14)$$

$$c_n = c_n(P) = \int_0^1 P(x) \overline{\psi_n(x)} dx , \quad \forall n \geq 0, \quad (2.15)$$

$$a_i = a_i(\chi_{\Delta_m^{(k)}}) , \quad 0 \leq i < a^m , \quad b_j = b_j(I_{\nu_0}^{(1)}) , \quad 1 \leq j < a^{\nu_0} . \quad (2.16)$$

Taking into account (2.1)-(2.3), (2.5)-(2.7), (2.9)-(2.12) for  $P(x)$  we obtain

$$P(x) = \gamma \cdot \sum_{i=0}^{a^m-1} a_i \psi_i(x) \cdot \sum_{j=1}^{a^{\nu_0}-1} b_j \psi_j(a^s x) = \quad (2.17)$$

$$= \gamma \cdot \sum_{j=1}^{a^{\nu_0}-1} b_j \cdot \sum_{i=0}^{a^m-1} a_i \psi_{j \cdot a^s + i}(x) = \sum_{k=N_0}^N c_k \psi_k(x),$$

where

$$c_k = c_k(P) = \begin{cases} -\mathcal{K} \cdot \frac{\gamma}{a^m} \text{ or } 0, & \text{if } k \in [N_0, N] \\ 0, & \text{if } k \notin [N_0, N], \end{cases} \quad (2.18)$$

$$\mathcal{K} \in \Omega_a, \quad |\mathcal{K}| = 1, \quad N = a^{s+\nu_0} + a^m - a^s - 1. \quad (2.19)$$

Set

$$E = \{x \in \Delta : P(x) = \gamma\}.$$

By (2.7), (2.8) and (2.14) we have

$$|E| = a^{-m}(1 - a^{-\nu_0}) > (1 - \epsilon)|\Delta|,$$

$$P(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ \gamma(1 - a^{\nu_0}), & \text{if } x \in \Delta \setminus E, \\ 0, & \text{if } x \notin \Delta. \end{cases}$$

Hence and from (2.13) we get

$$\int_0^1 |P(x)| dx = 2 \cdot |\gamma| |\Delta| \cdot (1 - a^{-\nu_0}),$$

and taking into account that  $a \geq 2$  we have

$$\frac{1}{2} \cdot |\gamma| \cdot |\Delta| < \int_0^1 |P(x)| dx < 2 \cdot |\gamma| \cdot |\Delta|.$$

From relations (2.13), (2.18) and (2.19) we obtain

$$\begin{aligned} \max_{N_0 \leq m \leq N} \int_0^1 \left| \sum_{k=N_0}^m c_k \psi_k(x) \right| dx &< \left[ \int_0^1 |P(x)|^2 dx \right]^{\frac{1}{2}} \leq \left[ \sum_{k=N_0}^N c_k^2 \right]^{\frac{1}{2}} = \\ &= |\gamma| \cdot |\Delta| \cdot \sqrt{a^{\nu_0+s} + a^m} = |\gamma| \cdot \sqrt{|\Delta|} \cdot \sqrt{a^{\nu_0} + 1} < \\ &< |\gamma| \cdot \sqrt{|\Delta|} \cdot \sqrt{\frac{a}{\epsilon}} < a \cdot |\gamma| \cdot \sqrt{\frac{|\Delta|}{\epsilon}}. \end{aligned}$$

□

**Lemma 2.2.** *For any given numbers  $N_0 > 1$ , ( $N_0 \in \mathcal{N}$ ),  $\epsilon \in (0, 1)$  and each function  $f(x) \in L^1[0, 1)$ ,  $\|f\|_1 > 0$  there exists a measurable set  $E \subset [0, 1)$ , function  $g(x) \in L^1[0, 1)$  and a polynomial by  $\Psi_a$  system of the form*

$$P(x) = \sum_{k=N_0}^N c_k \psi_{n_k}(x), \quad n_k \uparrow$$

satisfying the following conditions:

- 1)  $|E| > 1 - \epsilon,$
- 2)  $f(x) = g(x), \quad x \in E,$

$$3) \quad \frac{1}{2} \int_0^1 |f(x)| dx < \int_0^1 |g(x)| dx < 3 \int_0^1 |f(x)| dx.$$

$$4) \quad \int_0^1 |P(x) - g(x)| dx < \varepsilon.$$

$$5) \quad \varepsilon > |c_k| \geq |c_{k+1}| > 0.$$

$$6) \quad \max_{N_0 \leq m \leq N} \int_0^1 \left| \sum_{k=N_0}^m c_k \psi_{n_k}(x) \right| dx < 3 \int_0^1 |f(x)| dx.$$

*Proof.* Consider the step function

$$\varphi(x) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \cdot \chi_{\Delta_\nu}(x), \quad (2.20)$$

where  $\Delta_\nu$  are  $a$ -dyadic, not crossed intervals of the form  $\Delta_m^{(k)} = [\frac{k-1}{a^m}, \frac{k}{a^m})$ ,  $k = 1, 2, \dots, a^m$  so that

$$0 < |\gamma_\nu|^2 |\Delta_\nu| < \frac{\varepsilon^3}{16a^2} \cdot \left( \int_0^1 |f(x)| dx \right)^2. \quad (2.21)$$

$$0 < |\gamma_{\nu_0}| |\Delta_{\nu_0}| < \dots < |\gamma_\nu| |\Delta_\nu| < \dots < |\gamma_1| |\Delta_1| < \frac{\varepsilon}{2}. \quad (2.22)$$

$$\int_0^1 |f(x) - \varphi(x)| dx < \min\left\{\frac{\varepsilon}{4}; \frac{\varepsilon}{4} \int_0^1 |f(x)| dx\right\}. \quad (2.23)$$

Applying Lemma 2.1 successively, we can find the sets  $E_\nu \subset [0, 1)$  and a polynomial

$$P_\nu(x) = \sum_{k=N_{\nu-1}}^{N_\nu-1} c_k \psi_{n_k}(x), \quad 1 \leq \nu \leq \nu_0, \quad (2.24)$$

which, for all  $1 \leq \nu \leq \nu_0$ , satisfy the following conditions:

$$|c_k| = |\gamma_\nu| \cdot |\Delta_\nu|, \quad k \in [N_{\nu-1}, N_\nu) \quad (2.25)$$

$$|E_\nu| > (1 - \varepsilon) \cdot |\Delta_\nu|, \quad (2.26)$$

$$P_\nu(x) = \begin{cases} \gamma_\nu & : x \in E_\nu \\ 0 & : x \notin \Delta_\nu, \end{cases} \quad (2.27)$$

$$\frac{1}{2} |\gamma_\nu| \cdot |\Delta_\nu| < \int_0^1 |P_\nu(x)| dx < 2 |\gamma_\nu| \cdot |\Delta_\nu|. \quad (2.28)$$

$$\max_{N_{\nu-1} \leq m \leq N_\nu} \int_0^1 \left| \sum_{k=N_0}^m c_k \psi_{n_k}(x) \right| < a \cdot |\gamma_\nu| \cdot \sqrt{\frac{|\Delta_\nu|}{\varepsilon}}. \quad (2.29)$$

Define a set  $E$ , a function  $g(x)$  and a polynomial  $P(x)$  in the following way:

$$P(x) = \sum_{\nu=1}^{\nu_0} P_\nu(x) = \sum_{k=N_0}^N c_k \psi_{n_k}(x), \quad N = N_{\nu_0} - 1. \quad (2.30)$$

$$g(x) = P(x) + f(x) - \varphi(x). \quad (2.31)$$

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu. \quad (2.32)$$

From (2.20), (2.23), (2.26)-(2.28), (2.30)-(2.32) we have

$$\begin{aligned} |E| &> 1 - \varepsilon, \\ f(x) &= g(x), \quad \text{for } x \in E, \\ \frac{1}{2} \int_0^1 |f(x)| dx &< \int_0^1 |g(x)| dx < 3 \int_0^1 |f(x)| dx. \end{aligned}$$

By (2.22), (2.23), (2.25) and (2.31) we get

$$\int_0^1 |P(x) - g(x)| dx = \int_0^1 |f(x) - \varphi(x)| dx < \varepsilon.$$

$$\varepsilon > |c_k| \geq |c_{k+1}| > 0, \quad \text{for } k = N_0, N_0 + 1, \dots, N - 1.$$

That is, assertions 1)-5) of Lemma 2.2 actually hold. We now verify assertion 6). For any number  $m$ ,  $N_0 \leq m \leq N$  we can find  $j$ ,  $1 \leq j \leq \nu_0$  such that  $N_{j-1} < m \leq N_j$ . then by (2.24) and (2.30) we have

$$\sum_{k=N_0}^m c_k \psi_{n_k}(x) = \sum_{n=1}^{j-1} P_n(x) + \sum_{k=N_{j-1}}^m c_k \psi_{n_k}(x).$$

hence and from relations (2.21), (2.23), (2.28), (2.29) we obtain

$$\begin{aligned} \int_0^1 \left| \sum_{k=N_0}^m c_k \psi_{n_k}(x) \right| dx &\leq \sum_{\nu=1}^{\nu_0} \int_0^1 |P_\nu(x)| dx + \int_0^1 \left| \sum_{k=N_{j-1}}^m c_k \psi_{n_k}(x) \right| dx < \\ &< 2 \int_0^1 |\varphi(x)| dx + a \cdot |\gamma_j| \cdot \sqrt{\frac{|\Delta_j|}{\varepsilon}} < 3 \int_0^1 |f(x)| dx. \end{aligned}$$

□

### 3. MAIN RESULTS

*Proof.* Let

$$\{f_n(x)\}_{n=1}^\infty \quad (3.1)$$

be a sequence of all step functions, values and constancy interval endpoints of which are rational numbers. Applying Lemma 2.2 consecutively, we can find a sequences of functions  $\{\bar{g}_n(x)\}$  of sets  $\{E_n\}$  and a sequence of polynomials

$$\bar{P}_n(x) = \sum_{k=N_{n-1}}^{N_n-1} c_{m_k} \psi_{m_k}(x), \quad N_0 = 1, \quad |c_{m_k}| > 0 \quad (3.2)$$

which satisfy the conditions:

$$|E_n| > 1 - \varepsilon \cdot 4^{-8(n+2)} \quad (3.3)$$

$$f_n(x) = \bar{g}_n(x), \quad \text{for all } x \in E_n, \quad (3.4)$$



$$\frac{1}{2} \int_0^1 |f_n(x)| dx < \int_0^1 |\bar{g}_n(x)| dx < 3 \int_0^1 |f_n(x)| dx. \quad (3.5)$$

$$\int_0^1 |\bar{P}_n(x) - \bar{g}_n(x)| dx < 4^{-8(n+2)}. \quad (3.6)$$

$$\max_{N_{n-1} \leq M \leq N_n} \int_0^1 \left| \sum_{k=N_{n-1}}^M c_{m_k} \psi_{m_k}(x) \right| dx < 3 \int_0^1 |f_n(x)| dx. \quad (3.7)$$

$$\frac{1}{n} > |c_{m_k}| > |c_{m_{k+1}}| > |c_{m_{N_n}}| > 0. \quad (3.8)$$

Set

$$\sum_{k=1}^{\infty} c_{m_k} \psi_{m_k}(x) = \sum_{n=1}^{\infty} \bar{P}_n(x) = \sum_{n=1}^{\infty} \sum_{k=N_{n-1}}^{N_n-1} c_{m_k} \psi_{m_k}(x), \quad (3.9)$$

and

$$E = \bigcap_{n=1}^{\infty} E_n. \quad (3.10)$$

It is easy to see that (see (3.3)),  $|E| > 1 - \varepsilon$ .

Now we consider a series

$$\sum_{i=1}^{\infty} c_i \psi_i(x)$$

where  $c_i = c_{m_k}$   $i \in [m_k, m_{k+1})$ . From (3.8) it follows that  $|c_i| \downarrow 0$ .

Let given any function  $f(x) \in L^1[0, 1)$  then we can choose a subsequence  $\{f_{s_n}(x)\}_{n=1}^{\infty}$  from (3.1) such that

$$\lim_{N \rightarrow \infty} \int_0^1 \left| \sum_{n=1}^N f_{s_n}(x) - f(x) \right| dx = 0, \quad (3.11)$$

$$\int_0^1 |f_{s_n}(x)| dx \leq \epsilon \cdot 4^{-8(n+2)}, n \geq 2, \quad (3.12)$$

where

$$\epsilon = \min\left\{\frac{\varepsilon}{2}, \int_E |f(x)| dx\right\}. \quad (3.13)$$

We set

$$g_1(x) = \bar{g}_{s_1}(x), \quad P_1(x) = \bar{P}_{s_1}(x) = \sum_{k=N_{s_1-1}}^{N_{s_1}-1} c_{m_k} \psi_{m_k}(x) \quad (3.14)$$

It is easy to see that

$$\int_0^1 |f(x) - f_{k_1}(x)| < \frac{\epsilon}{2}$$

Taking into account (3.5), (3.7) and (3.14) we have

$$\max_{N_{s_1-1} \leq M \leq N_{s_1}} \int_0^1 \left| \sum_{k=N_{s_1-1}}^M c_{m_k} \psi_{m_k}(x) \right| dx < 3 \int_0^1 |f_{s_1}(x)| dx < 6 \int_0^1 |g_1(x)| dx.$$

Then assume that numbers  $\nu_1, \nu_2, \dots, \nu_{q-1}$  ( $\nu_1 = s_1$ ), functions  $g_n(x)$ ,  $f_{\nu_n}(x)$ ,  $n = 1, 2, \dots, q-1$  and polynomials

$$P_n(x) = \sum_{k=M_n}^{\overline{M}_n} c_{m_k} \psi_{m_k}(x), \quad M_n = N_{\nu_n-1}, \quad \overline{M}_n = N_{\nu_n} - 1,$$

are chosen in such a way that the following condition is satisfied:

$$g_n(x) = f_{s_n}(x), \quad x \in E_{\nu_n}, \quad 1 \leq n \leq q-1, \quad (3.15)$$

$$\int_0^1 |g_n(x)| dx < 4^{-3n} \epsilon, \quad 1 \leq n \leq q-1, \quad (3.16)$$

$$\int_0^1 \left| \sum_{k=2}^n (P_k(x) - g_k(x)) \right| dx < 4^{-8(n+1)} \epsilon, \quad 1 \leq n \leq q-1, \quad (3.17)$$

$$\max_{M_n \leq M \leq \overline{M}_n} \int_0^1 \left| \sum_{k=M_n}^M c_{m_k} \psi_{m_k}(x) \right| dx < 4^{-3n} \epsilon, \quad 1 \leq n \leq q-1. \quad (3.18)$$

We choose a function  $f_{\nu_q}(x)$  from the sequence (3.1) such that

$$\int_0^1 \left| f_{\nu_q}(x) - \left[ f_{s_q}(x) - \sum_{k=2}^n (P_k(x) - g_k(x)) \right] \right| dx < 4^{-8(q+2)} \epsilon. \quad (3.19)$$

This with (3.11) imply

$$\int_0^1 \left| f_{\nu_q}(x) - \sum_{k=2}^n (P_k(x) - g_k(x)) \right| dx < 4^{-8q-1} \epsilon,$$

and taking into account relation (3.19) we get

$$\int_0^1 |f_{\nu_q}(x)| dx < 4^{-8q} \epsilon. \quad (3.20)$$

We set

$$P_q(x) = \overline{P}_{\nu_q}(x) = \sum_{k=M_q}^{\overline{M}_q} c_{m_k} \psi_{m_k}(x), \quad (3.21)$$

where

$$\begin{aligned} M_q &= N_{\nu_q-1}, \quad \overline{M}_q = N_{\nu_q} - 1, \\ g_q(x) &= f_{s_q}(x) + [\overline{g}_{\nu_q}(x) - f_{\nu_q}(x)] \end{aligned} \quad (3.22)$$

By (3.4)-(3.7), (3.17)-(3.22) we have

$$g_q(x) = f_{s_q}(x), \quad x \in E_{\nu_q}, \quad (3.23)$$

$$\begin{aligned} & \int_0^1 |g_q(x)| dx \leq \\ & \leq \int_0^1 \left| f_{\nu_q}(x) - \left[ f_{s_q}(x) - \sum_{k=2}^n (P_k(x) - g_k(x)) \right] \right| dx + \end{aligned} \quad (3.24)$$

$$+ \int_0^1 |\bar{g}_{\nu_q}(x)| dx + \int_0^1 \left| \sum_{k=2}^n (P_k(x) - g_k(x)) \right| dx < 4^{-3n} \epsilon,$$

$$\int_0^1 \left| \sum_{k=2}^q (P_k(x) - g_k(x)) \right| dx \leq \quad (3.25)$$

$$\leq \int_0^1 \left| f_{\nu_q}(x) - \left[ f_{s_q}(x) - \sum_{k=2}^n (P_k(x) - g_k(x)) \right] \right| dx +$$

$$+ \int_0^1 |\bar{P}_{\nu_q}(x) - \bar{g}_{\nu_q}(x)| dx < 4^{-8(n+1)} \epsilon,$$

$$\max_{M_q \leq M \leq \bar{M}_q} \int_0^1 \left| \sum_{k=M_q}^M c_{m_k} \psi_{m_k}(x) \right| dx \leq 3 \int_0^1 |f_{\nu_q}(x)| dx < 4^{-3n} \epsilon. \quad (3.26)$$

Thus, by induction we can choose the sequences of sets  $\{E_q\}$ , functions  $\{g_q(x)\}$  and polynomials  $\{P_q(x)\}$  such that conditions (3.23) - (3.26) are satisfied for all  $q \geq 1$ . Define a function  $g(x)$  and a series in the following away:

$$g(x) = \sum_{n=1}^{\infty} g_n(x), \quad (3.27)$$

$$\sum_{n=1}^{\infty} \delta_n c_n \psi_n(x) = \sum_{n=1}^{\infty} \left[ \sum_{k=M_n}^{\bar{M}_n} c_{m_k} \psi_{m_k}(x) \right], \quad (3.28)$$

where

$$\delta_n = \begin{cases} 1, & \text{if } i = m_k, \text{ where } k \in \bigcup_{q=1}^{\infty} [M_q, \bar{M}_q] \\ 0, & \text{in the other case.} \end{cases}$$

Hence and from relations (3.5), (3.10), (3.15), (3.27),

$$g(x) = f(x), \quad x \in E, \quad g(x) \in L^1[0, 1), \quad (3.29)$$

$$\frac{1}{2} \int_0^1 |f(x)| dx < \int_0^1 |g(x)| dx < 4 \int_0^1 |f(x)| dx. \quad (3.30)$$

Taking into account (3.21), (3.24)-(3.28) we obtain that the series (3.28) convergence to  $g(x)$  in  $L^1[0, 1)$  metric and consequently is its Fourier series by  $\Psi_a$  system,  $a \geq 2$ .

From Definition 1.3, and from relations (3.13), (3.18), (3.30) for any natural number  $m$  there is  $N_m$  so that

$$\|G_m(g)\|_1 = \|S_m(g)\|_1 = \int_0^1 \left| \sum_{n=1}^{\infty} \delta_n c_n \psi_n(x) \right| dx \leq 4 \int_0^1 |f(x)| dx$$

$$\begin{aligned}
&\leq \sum_{n=1}^{\infty} \left( \max_{M_n \leq M \leq \overline{M}_n} \int_0^1 \left| \sum_{k=M_n}^M c_{m_k} \psi_{m_k}(x) \right| dx \right) \leq \\
&\leq 2 \int_0^1 |g_1(x)| dx + \epsilon \cdot \sum_{n=2}^{\infty} 4^{-n} \leq \\
&\leq 3 \int_0^1 |g(x)| dx \leq 12 \int_0^1 |f(x)| dx = 12 \|f\|_1.
\end{aligned}$$

□

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